# Dying on Birthdays

A fixture of life contingencies is the following definition of the notation ${}_tq_x$:

$${}_tq_x = F_X(t) = \text{Prob}[T(x) \le t] ,$$

where $T(x)$ is the complete-future-life random variable for $(x)$ – a life aged exactly $x$. $F_X$ is the cumulative distribution function of $T(x)$. X is the age-at-death random variable, so $F_X(u)=0$ for $u \le x$. (This is the notation of Bowers *et al*.)

In practice, however, this is not the definition we use for ${}_tq_x$ . The definition we use is

$${}_tq_x = \text{Prob}[T(x) < t] .$$

Note the strict inequality. That this is the definition we use becomes clear when we verbalize ${}_tq_x$ as

(a) "the probability $(x)$ will not survive $t$ years," and
(b) "the probability $(x)$ will be dead $t$ years from now."

The discrepancy between Prob[$T(x) \le t$] and our verbalizations for ${}_tq_x$ is most pointed when both $x$ and $t$ are integers. In that case, "now" is the $x^{th}$ birthday of the life, and we are interested in the $(x+t)^{th}$ birthday. Then we verbalize (paraphrase) ${}_tq_x$ as follows:

(c) "the probability $(x)$ will not survive until his $(x+t)^{th}$ birthday," and
(d) "the probability that $(x)$ will die before his $(x+t)^{th}$ birthday."

If we were using the definition

$${}_tq_x = F_X(t) = \text{Prob}[T(x) \le t]$$

these verbalizations would be that ${}_tq_x$ is

(a′) "the probability $(x)$ will not survive more than $t$ years," and
(b′) "the probability that at any time later than $t$ years from now $(x)$ will be dead," and
(c′) "the probability $(x)$ does not survive past his $(x+t)^{th}$ birthday," and
(d′) "the probability that $(x)$ will die on or before his $(x+t)^{th}$ birthday."

This discrepancy between our definition and our practice is worrisome in view of how we calculate the EPV of term assurances and endowments.

Suppose on his $x^{th}$ birthday the life purchased both a $t$-year term assurance and a $t$-year pure endowment. In our calculations, we note that exactly one of these contracts will pay, and we take ${}_tq_x$ to be the probability that the term assurance will pay and ${}_tp_x$ to be the probability the pure endowment pays. In the case that the life dies on his $(x+t)^{th}$ birthday,

(a) our practice is that the pure endowment will pay and the term assurance will not pay, but

(b) if we were using ${}_tq_x = F_X(t) = \text{Prob}[T_x \le t]$, the term assurance would pay and the pure endowment would not pay.

This is not a problem of semantics. Our practice and our verbalizations are in agreement. They agree on the fact that if death takes place on a birthday, then the life survived to that birthday, and death did not occur before the birthday.

In principle, the range of $T(x)$, and the domain of $F_X$, is the continuum $(0, \omega-x)$. In practice, however, birthdates and death dates are known, but birth times and death times are not known, so the range of $T(x)$, and the domain of $F_X$, is made discrete by letting $t$ increase each 24 hours, taking on 364 values between consecutive integers. Giving $t$ in years, with three decimal places, will distinguish these 364 values from each other.

This discretization does not, however, resolve the conflict with our verbalizations. If (41) dies on his $64^{th}$ birthday, we will say he survived to his $64^{th}$ birthday, and $T(41) = 23.000$.

The remedy is to introduce a new function, $G_X(t) \equiv \text{Prob}[T(x) < t]$, and to define ${}_tq_x$ as

$${}_tq_x \equiv G_X(t) = \text{Prob}[T(x) < t].$$

$G_X$ is not a cumulative distribution function, but it is differentiable to the same extent as $F_X$ is differentiable. Its derivative, $g_X \equiv G'_X$, is not a density function, but $g_X$ has all the properties we need.

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