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Determination of the Value at Risk

Using Approximate Methods

RECENT WORK BY VANDERHOOF, ALBERT, AND LORD USED SIMULATION METHODOLOGY to determine the probability distribution of the total loss incurred on a group of insurance policies. The simulation methodology was based on low discrepancy technology, which results in a more efficient use of the computer as opposed to the traditional Monte Carlo technique. Before the advent of more powerful computing aids, approximate methods using theoretical probability distributions were used. The accuracy of these methods depends upon many factors, such as the number of policies, the mortality rates, and the variability of the amount of insurance coverage from policy to policy.

The advantage of using these approximate methods is the ease of computation. When the number of policies becomes large, the low discrepancy simulation methodology discussed by Vanderhoof, Albert, and Lord and the exact methods using the convolution approach, discussed in the text on loss models by Klugman, Panjer, and Wilmot may become onerous. Of course accuracy is an important issue.

In this article we will examine three possible approximations for computing the probability of exceeding a given loss level, namely the normal approximation, the lognormal approximation, and the gamma distribution. An excellent review of the characteristics of these probability distributions appears in Johnson and Kotz and Klugman, Panjer, and Wilmot.

Data Used

Vanderhoof, Albert, and Lord used a group of 10,000 policies to compute the probability of not exceeding given levels of claims under different scenarios of maximum retention levels and mortality levels. Here, we'll use the same group of policies and examine three scenarios. In this way we'll compare the approximate probabilities with the probabilities estimated from the low discrepancy simulation approach. (See Table 1.)

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For each of these scenarios, the probability distribution of the total claims will be first approximated by using the normal distribution. The justification for the use of the normal distribution is the Central Limit Theorem, which states that the probability distribution of the sum of a set of independent random variables will tend, in the limit, to a normal distribution if certain conditions are met. In this case the number of variables, namely 10,000 policies, is not sufficiently large for this approximation to work well. The actual claims distributions of a total of 10,000 policies will be positively skewed, which will make this approximation understate the true probabilities of exceeding a relatively large claim level. The other two approximations, namely the lognormal and gamma distributions, are positively skewed probability distributions. These latter two approximations will provide better results.

Each of these three probability distributions consists of two parameters. For each scenario in Table 1 and for each approximation, the claims distribution will be approximated by equating the means and standard deviations in Table 1 to the means and standard deviations for the corresponding probability distribution. We can then solve the resulting equations to determine the corresponding values of the unknown parameters. Following this computation, we can calculate the approximate probabilities by integration of the corresponding probability density functions.

These approximations are based in the individual risk theory model. Other approximations have been developed using the collective risk theory model. (A discussion of these two models appears in Klugman, Panjer, and Wilmot.)

Results

We'll present the results of our computations in two ways. First, we'll compare the probabilities of exceeding a given level of claims for each of the three scenarios, using the three approximations, with the simulation results of Vanderhoof, Albert, and Lord. We'll also compare the percentiles (50 percent, 90 percent, 95 percent, 99 percent, and 99.9 percent).

Table 2A compares the probabilities for scenario A. The normal approximation consistently underestimates the probability for relatively high claim levels. The claims distribution is too skewed to make the normal distribution a viable approximation in the right tails of the distribution (the upper 5 percent). The normal approximation would be viable only if the number of policies were much larger. The lognormal and gamma distributions provide better results, and in particular the lognormal distributions provide the best results for the upper 5 percent of the curve. This happens because the lognormal distribution has a higher skewness coefficient than the gamma distribution. The skewness level of the lognormal is probably closer to the skewness level of the claims probability distribution. (The coefficient of skewness for the gamma distribution equals 2Φ / :, whereas the coefficient of skewness for the lognormal distribution equals 2Φ / : + (Φ / :)3, where : and Φ are respectively the mean and standard deviation of the corresponding distributions.)

Table 2B evaluates the approximation under the assumption that the mortality levels are doubled but the retention level remains the same as scenario A. The same trends occur for this scenario. The lognormal approximation performs better than the gamma for claim levels in the upper 5 percent of the claims curve.

Table 2C evaluates the approximation under the assumption that the retention level is reduced to $75,000, with mortality remaining at the same level as in scenario A. For this scenario, the gamma distribution performs better for the upper 5 percent of the claims distribution. This trend occurs because the claims distribution will become less skewed with a lower retention level. The lognormal has a

TABLE 1 Scenarios Examined

SCENARIO	MAXIMUM RETENTION	MORTALITY BASIS	MEAN CLAIM	STANDARD DEVIATION
A	$3,200,000	Individual Annuity 2000	$1,391,205	$589,269
B	3,200,000	Individual Annuity 2000 (Double)	2,782,410	832,112
C	75,000	Individual Annuity 2000	792,241	234,422

TABLE 2A Comparison of Probabilities of Exceeding a Given Claim Level—for Scenario A

		APPROXIMATIONS		
CLAIM LEVEL	SIMULATION	NORMAL	LOGNORMAL	GAMMA
$1,301,150	50.00%	56.07%	48.47%	50.55%
1,903,750	15.87	19.22	16.47	17.90
2,124,240	10.00	10.68	10.66	11.29
2,543,980	5.00	2.52	4.56	4.29
2,845,660	2.28	0.68	2.47	2.02
3,280,750	1.00	0.07	1.03	0.64
4,525,890	0.10	0.00	0.09	0.02

Maximum retention = $3,200,000; Mortality basis is individual annuity 2000; Mean claim = $1,391,205; Standard deviation = $589,269

TABLE 2B Comparison of Probabilities of Exceeding a Given Claim Level—for Scenario B

		APPROXIMATIONS		
CLAIM LEVEL	SIMULATION	NORMAL	LOGNORMAL	GAMMA
$2,677,700	50.00%	55.01%	49.39%	51.09%
3,577,960	15.87	16.95	15.73	16.47
3,888,940	10.00	9.18	9.85	9.98
4,314,390	5.00	3.28	5.00	4.67
4,765,830	2.28	0.86	2.36	1.92
5,267,250	1.00	0.14	1.00	0.65
6,385,130	0.10	0.00	0.14	0.05

Maximum retention = $3,200,000; Mortality basis is double that of individual annuity 2000; Mean claim = $2,782,410; Standard deviation = $831,112

TABLE 2C Comparison of Probabilities of Exceeding a Given Claim Level—for Scenario C

		APPROXIMATIONS		
CLAIM LEVEL	SIMULATION	NORMAL	LOGNORMAL	GAMMA
$ 775,700	50.00%	52.80%	47.11%	48.87%
1,034,010	15.87	15.12	14.36	14.95
1,092,140	10.00	10.04	10.51	10.71
1,211,230	5.00	3.69	5.37	5.06
1,315,820	2.28	1.28	2.89	2.46
1,394,440	1.00	0.51	1.80	1.38
1,664,450	0.10	0.01	0.34	0.16

Maximum retention = $75,000; Mortality basis is individual annuity 2000; Mean claim = $792,241; Standard deviation = $234,422

TABLE 3A Comparison of Percentiles for the Claims Distribution—for Scenario A

PERCENTILE	SIMULATION	APPROXIMATIONS NORMAL	LOGNORMAL	GAMMA
90	$2,124,240	$2,146,384 (1.0)	$2,155,998 (1.5)	$2,179,467 (2.6)
95	2,543,980	2,360,466 (–7.2)	2,498,858 (–1.7)	2,480,437 (–2.5)
99	3,280,750	2,762,050 (–16)	3,295,868 (0.4)	3,113,292 (–5.1)
99.9	4,525,890	3,212,183 (–29)	4,495,837 (–0.7)	3,932,241 (–3.1)

Maximum retention = $3,200,000; Mortality basis is individual annuity 2000; Mean claim = $1,391,205; Standard deviation = $589,269. Numbers in parentheses represent % difference between the approximate percentiles and percentiles based on simulation.

TABLE 3B Comparison of Percentiles for the Claims Distribution—for Scenario B

PERCENTILE	SIMULATION	APPROXIMATIONS NORMAL	LOGNORMAL	GAMMA
90	$3,888,940	$3,847,523 (–1.1)	$3,877,683 (–0.3)	$3,886.439 (0.0)
95	4,314,390	4,149,468 (–3.8)	4,312,193 (0.0)	4,275,445 (–0.9)
99	5,267,250	4,715,866 (–10)	5,262,871 (0.0)	5,070,753 (–3.7)
99.9	6,385,130	5,350,739 (–16)	6,579,725 (3.0)	6,066,892 (–5.0)

Maximum retention = $3,200,000; Mortality basis is double that of individual annuity 2000; Mean claim = $2,782,410; Standard deviation = $831,112. Numbers in parentheses represent % difference between the approximate percentiles and percentiles based on simulation.

TABLE 3C Comparison of Percentiles for the Claims Distribution—for Scenario C

PERCENTILE	SIMULATION	APPROXIMATIONS NORMAL	LOGNORMAL	GAMMA
90	$1,092,140	$1,092,665 (0.0)	$1,101,225 (0.8)	$1,103,576 (1.0)
95	1,211,230	1,177,831 (–2.8)	1,223,451 (1.0)	1,213,071 (0.2)
99	1,394,440	1,337,588 (–4.1)	1,490,501 (6.9)	1,436,788 (3.0)
99.9	1,664,450	1,516,659 (–8.8)	1,859,704 (11.7)	1,716,753 (3.1)

Maximum retention = $75,000; Mortality basis is individual annuity 2000; Mean claim = $792,241; Standard deviation = $234,442. Numbers in parentheses represent % difference between the approximate percentiles and percentiles based on simulation.

higher level of skewness than the claims distribution. The gamma approximation performs better for this scenario because the gamma distribution's level of skewness is more in line with the skewness level of the claims distribution.

Table 3A compares the percentiles for scenario A. As an example, the 99th percentile for the claims distribution is $3,280,750 when using the simulation results. This implies that there is a 90 percent probability that the actual loss will be less than this number and a 10 percent probability of exceeding this number. If the lognormal approximation is used, this percentile will become $3,295,868, which is 0.4 percent higher than the simulation result.

The results of Tables 3A and 3B also show that for scenarios A and B, the lognormal provides a better approximation. The corresponding percentiles are within 3 percent of the simulation results. These two scenarios represent a higher retention level and a higher level of skewness. Table 3C shows that for scenario C, a lower retention level and a lower skewness level, the gamma provides a better approximation. These results are consistent with the results of Tables 2A, 2B, and 2C.

Conclusions

We examined three approximations to the probability distributions of claims under a group of 10,000 policies. The normal distribution wasn't a viable alternative, because the claims probability distribution was too skewed. This resulted in an underestimation of the probability of exceeding a relatively high claims level and an underestimation of the higher percentiles.

The other two approximations, namely the lognormal and the gamma, performed better. The lognormal tends to be more heavily skewed and was the better approach when dealing with a more heavily skewed claims distribution, whereas the gamma was the better approach when dealing with a less heavily skewed claims distribution.

The basic problem is that the three approximations were based on a two-parameter distribution. In order to model the level of skewness, probability distributions that have three or more parameters are needed. We are currently researching possible solutions to this problem, which include the Edgeworth approximation and the Pearson type-3 distribution. ●

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