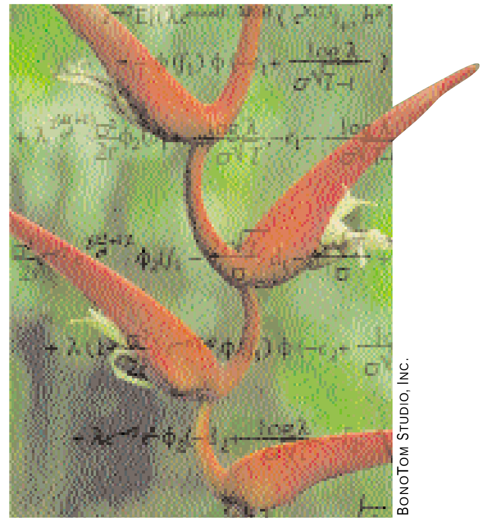
BY HANGSUCK LEE

Pricing Equity-Indexed Annuities Embedded with Exotic Options



BONOTOM STUDIO, INC.

IN THE DEFERRED ANNUITIES MARKET, THE PORTION OF FIXED-RATE ANNUITIES in annual sales has declined from one-half in 1994 to one-quarter in 1999. This is due in part to relatively low interest rates and a bullish stock market. The new economic environment has led actuaries to design new types of annuity products that link return to stock market performance. One of these products is equity-indexed annuities (EIAs). When a stock index, typically the S&P 500, goes up, EIAs provide policyholders with a rate of return connected to the return of the index. When the index goes down, EIAs provide policyholders with a minimum guaranteed return.

Since the first offering in 1995, sales of EIAs have rapidly increased. In 1996, 1997, 1998, 1999, and 2000, sales were at $1.5, $3.0, $4.3, $5.1, and $5.4 billion, respectively.

There are several reasons for the increasing popularity of EIAs. First, they offer returns that are linked to the performance of an equity index and never fall lower than a minimum guaranteed return (usually 3 percent per annum). EIAs may be the most suitable products for those who normally are reluctant to buy traditional fixed annuities because of low returns and are also reluctant to buy mutual funds and stocks for fear of high volatility in the stock market.

Second, EIAs may serve as tax deferral investment vehicles and possess a tax advantage relative to mutual funds and other bank products.

Third, EIAs also appeal to insurance companies and agents because companies selling them don't have to register with the Securities and Exchange Commission (SEC) and agents don't need a special license to sell them.

However, pricing, hedging, and reserving EIAs are challenging problems due to the complex payoff structure. Although insurers have experience in managing interest risk and mortality risk, the minimum guarantees related to the equity market are relatively new to them. Unlike traditional life products, EIAs cannot be hedged through diversification. First of all, pricing EIAs is the prerequisite for hedging and reserving them. Evaluating the guarantee embedded in an EIA is difficult and often requires advanced stochastic modeling techniques.

Two key factors for pricing EIAs are participation rate and indexing method. The participation rate is the percentage of the index return to be credited. Thus, the insurance company credits the EIA policy either the index return times the participation rate or a minimum guaranteed return, whichever is greater.

For example, if we assume that the index return is 20 percent, the participation rate is 90 percent, and the minimum guaranteed return is 15 percent, then the actual return credited to the policy will be 18 percent.

An indexing method is the method the insurer uses to calculate the index return based on the index values in the contract term. There are several indexing methods such as point-to-point, Asian, lookback, and annual reset. A point-to-point EIA, for example, measures the index increment from the start to the end of a contract term. The increment divided by the start index is the index return.

As pointed out in Tiong's paper "Valuing Equity-Indexed Annuities," sales of EIAs rapidly increased since the first offering, but growth rate has recently shown signs of slowing down because the current volatile equity market increases the costs of guarantees in the EIAs and hence decreases the participation rates. Thus, new EIAs with higher participation rates need to be designed EIAs that are similar to existing ones and have a cheaper guarantee such as that provided by a barrier option or a partial lookback option.

To make EIAs more attractive to customers, I propose three types of EIAs with higher participation rates: a barrier EIA, an annual reset EIA with barrier, and a partial lookback EIA. In addition, I shall pre-

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sent pricing formulas for them, using the *method of Esscher transforms* under a Black-Scholes framework.

Probability Distributions and Esscher Transforms

The payoffs of the proposed EIAs depend on the underlying index value and the maximum index value for a specified term. This section describes joint probability distributions and the method of Esscher transforms necessary for pricing them.

Let $S(t)$ denote the time-t price of an equity index. Assume that the index is constructed with all dividends reinvested. Assume that for $t \geq 0$,

$$S(t) = S(0)e^{X(t)}$$

where $\{X(t)\}$ is a Brownian motion with drift μ, diffusion coefficient σ and $X(0) = 0$. Thus $X(t)$ has a normal distribution with mean μt and variance $\sigma^2 t$. Let

$$M(t) = \max\{X(\tau), 0 \leq t \leq t\} \quad (2.1)$$

be the maximum of the Brownian motion between time 0 and time t. It is known that for $x \leq$ m and m ≥ 0,

$$\Pr(X(t) \leq x, M(t) \leq m = \Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2\mu}{\sigma^2}m}\,\Phi\left(\frac{x-2m-\mu t}{\sigma\sqrt{t}}\right) \quad (2.2)$$

where $\phi_2(.)$ denotes the standard normal distribution function.

Furthermore, in pricing the partial lookback and barrier EIAs, it's useful to know the joint distribution function of random variables $M(t)$ and $X(T)$, $0 < t \leq T$,

$$\Pr(M(t) \leq m, X(T) \leq x) \quad (2.3)$$

$$= \Phi_2\left(\frac{x-\mu T}{\sigma\sqrt{T}}, \frac{m-\mu t}{\sigma\sqrt{t}}, \sqrt{\frac{t}{T}}\right) - e^{\frac{2\mu}{\sigma^2}m}\,\Phi_2\left(\frac{x-2m-\mu T}{\sigma\sqrt{t}}, \frac{m-\mu t}{\sigma\sqrt{t}}, \sqrt{\frac{t}{T}}\right),$$

where $\Phi_2(a, b; \rho)$ denotes the bivariate standard normal distribution function.

Finally, Gerber and Shiu (1994, 1996) developed a special case of the method of Esscher transforms. For a nonzero real number h, the moment generating function of $X(t)$, $E[e^{hX(t)}]$ exists for all $t \geq 0$, because $\{X(t)\}$ is the Brownian motion as described above. The stochastic process

$$\{e^{hX(t)}E[e^{hX(t)}]^{-1}\}$$

is a positive martingale, which can be used to define a new probability measure Q. In technical terms, the process is used to define the Radon-Nikodym derivative dQ/dP, where P is the original probability measure. We call Q the Esscher measure of parameter h.

For a random variable Y that is a real-valued function of $\{X(t), 0 \leq t \leq T\}$, the expectation of Y under the new probability measure Q is calculated as

$$E\left[Y\frac{e^{hX(T)}}{E[e^{hX(T)}]}\right] \quad (2.4)$$

which will be denoted by $E[Y; h]$. Note that the process $\{X(t)\}$ under the Esscher measure of parameter h is a Brownian motion with drift $\mu + h\sigma^2$ and diffusion coefficient σ. The risk-neutral Esscher measure is the Esscher measure of parameter $h = h^*$ with respect to which the process $\{e^{-rt}S(t), 0 \leq t \leq T\}$ is a martingale. Thus,

$$E[e^{-rt}S(t); h^*] = S(0). \quad (2.5)$$

Therefore, h^* is the solution of

$$\mu + h^*\sigma^2 = r - \sigma^2/2. \quad (2.6)$$

Now, let us consider a special case of the factorization formula (Gerber and Shiu, 1994, p 177, 1996, p 188). For a random variable Y that is a real-valued function of $\{X(t), 0 \leq t \leq T\}$, the formula can be derived as follows:

$$E[e^{X(T)}Y; h^*] = e^{rT}\,E[Y; h^* + 1]. \quad (2.7)$$

The formula (2.7) will simplify many calculations in pricing the proposed EIAs.

Barrier EIA

The payoff of a point-to-point EIA depends only on the price of an underlying index on the ending date of the policy. The advantages of the EIA are its simplicity, making it relatively easy to explain the payoff to agents and customers, and good performance in a bullish equity market. However, the payoff will be the same regardless of the path taken by the index to attain its final value. If customers believe that the underlying index will rise above a barrier in a certain period, then they may not want to pay a high premium for the option embedded in a point-to-point EIA and will be reluctant to buy one.

The payoffs of barrier options are the same as those of their underlying plain-vanilla options if the path of an underlying asset satisfies an activating condition. Otherwise, they'll be zero. The barrier options are cheaper than their underlying plain-vanilla options because the payoffs of the barrier options are less than or equal to those of the plain-vanilla options.

In addition, sellers of barrier options may be able to limit their risk if they fail in hedging the options, because the payoffs are less volatile than those of their underlying plain-vanilla options. Barrier options have been substantially traded in the over-the-counter market since the late 1980s because they're attractive to both buyers and sellers.

To increase the participation rate, let us now apply an up-and-in barrier option to new EIAs as alternatives to point-to-point EIAs. The payoff of a barrier EIA is as follows:

If the index rises above a barrier in a certain period, the return credited to the policy will be the greater of either *a minimum guaranteed return or the index return times the participation rate*. If the index doesn't rise, the minimum guaranteed return will be credited. The cost of the option embedded in an EIA is key to increasing the participation rate; the cheaper the option price, the higher the participation rate and therefore the greater the customer's return. Thus, the barrier EIA provides customers with a higher participation rate than point-to-point EIAs.

Let's take a close look at the payoff of the barrier EIA with the monitoring period from time 0 to time t ($t < T$). Assume that the minimum guaranteed return is g for the contract term, the participation rate is α), and the barrier is B. Let $u = \log[B/S(0)]$ and $k = \log(1 + g/\alpha)$ Then the payoff can be expressed as follows:

$$\begin{cases} S(0)[1 + \alpha(e^{X(T)} - 1)], & \text{if } X(T) > k \text{ and } M(t) > u \\ S(0)(1 + g), & \text{otherwise,} \end{cases} \quad (3.1)$$

that is,

$$S(0)[\alpha(e^{X(T)} - \alpha - g)I(X(T) > k, M(t) > u) + (1 + g)], \quad (3.2)$$

where $I(\cdot)$ denotes the indicator function.

By the *fundamental theorem of asset pricing*, the time-0 value of the payoff (3.2) is

$$\begin{aligned} &S(0)e^{-rT}E[(\alpha e^{X(T)} - \alpha - g)I(X(T) > k, M(t) > u) + (1 + g); h^*] \\ &= S(0)[\alpha\, e^{-rT}E[e^{X(T)}I(X(T) > k, M(t) > u); h^*] \\ &- (\alpha + g)e^{-rT}\Pr(X(T) > k, M(t) > u; h^*) + e^{-rT}(1+g)]. \end{aligned} \quad (3.3)$$

It follows from (2.3) that we have the probability,

$$\begin{aligned} &\Pr(X(T) > k, M(t) > u) \\ &= \Phi_2\left(-\frac{k-\mu T}{\sigma\sqrt{T}}, \frac{-u+\mu t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}\right) + e^{\frac{2\mu}{\sigma^2}u}\,\Phi_2\left(-\frac{k-2u-\mu T}{\sigma\sqrt{T}}, \frac{-u-\mu t}{\sigma\sqrt{t}}; -\sqrt{\frac{t}{T}}\right) \\ &=: P_1(\mu). \end{aligned} \quad (3.4)$$

Applying the factorization formula (2.7) to the first expectation on the right-hand side of (3.3), the time-0 value of the barrier EIA is

$$S(0)[\alpha P_1(r + \sigma^2/2) - (\alpha + g)e^{-rT}P_1(r - \sigma^2/2) + e^{-rT}(1+g)]. \quad (3.5)$$

In particular, the time-0 value of the barrier EIA with the monitoring period from time 0 to time T is

$$S(0)[\alpha P_2(r + \sigma^2/2, T) - (\alpha + g)e^{-rT}P_2(r - \sigma^2/2, T) + e^{-rT}(1+g)] =: V(T), \quad (3.6)$$

where

$$P_2(\mu, T) := \Phi\left(\frac{-u-\mu T}{\sigma\sqrt{T}}\right) + e^{\frac{2\mu}{\sigma^2}u}\left[\Phi\left(\frac{-u-\mu T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{k-2u-\mu T}{\sigma\sqrt{T}}\right)\right].$$

Annual Reset EIA with Barrier

Unlike point-to-point EIAs, annual reset EIAs measure the annual returns on the index in each policy year and credit the policy annually compounded interest that is the greater of either the annual index return times the participation rate or a minimum guaranteed rate. These EIAs enable customers to profit in years the index goes up and never to lose in years the index goes down. Thus, the annual reset EIAs credit better interest to the policy in a volatile market.

A drawback of annual reset EIAs is that the participation rate is relatively low because these EIAs have one embedded option in each policy year. (For numerical values of the participation rate, see Tiong, 2000a, p. 44.) Though actuaries have developed annual reset EIAs with a cap or a spread to increase the participation rate, these EIAs have a disadvantage in that relatively low interest is credited to the policy when the index goes up. Just as we previously applied barrier options to point-to-point EIAs, let us now apply barrier options to annual reset EIAs to increase the participation rates.

In each period, the annual reset EIA will provide customers with the greater of either the annual index return times the participation rate or a minimum guaranteed rate if the maximum index value for each period rises above a barrier. Otherwise, the EIA will credit to the policyholder the minimum guaranteed rate as an annual return.

Note that the total return is defined as the return plus one (Luenberger, 1998, p. 138). Let

$$M_i = \max\{X(s) - X(t_{i-1}), t_{i-1} \le s \le t_i\}, \ i = 1, 2, \ldots, n \quad (4.1)$$

where $t_i - t_{i-1} = T/n$, $t_0 = 0$ and $t_n = T$. Assume that the minimum guaranteed rate is g for each period from time t_{i-1} to time t_i, and the participation rate is α for each period. Let $k = \log(1 + g/\alpha)$ and $u \ge k$. Assume $X_i = X(t_i) - X(t_{i-1})$. Hence, the random vectors (X_i, M_i) are independently and identically distributed with the random vector $(X(T/n), M(T/n))$. The total return of each period is as follows:

$$\begin{cases} 1 + \alpha(e^{X_i} - 1), & \text{if } X_i > k \text{ and } M_i > u \\ 1 + g, & \text{otherwise,} \end{cases} \quad (4.2)$$

that is,

$$(\alpha e^{X_i} - \alpha - g)I(X_i > k, M_i > u) + (1 + g). \quad (4.3)$$

Thus, the payoff of the annual reset EIA with barrier is

$$S(0)\prod_{i=1}^{n}[(\alpha e^{X_i} - \alpha - g)I(X_i > k, M_i > u) + (1 + g)]. \quad (4.4)$$

By the fundamental theorem of asset pricing, the time-0 value of the payoff (4.4) is calculated as follows:

$$S(0)e^{-rT}\mathrm{E}[\prod_{i=1}^{n}[(\alpha e^{X_i} - \alpha - g)\mathrm{I}(X_i > k, M_i > u) + (1 + g)]; h^*]$$

$$= S(0)\prod_{i=1}^{n} e^{-rT/n}\mathrm{E}[(\alpha e^{X_i} - \alpha - g)\mathrm{I}(X_i > k, M_i > u) + (1 + g); h^*]$$

$$= S(0)[V(T/n)]^n. \qquad (4.5)$$

Partial Lookback EIA

A lookback EIA credits the policy the interest based on the maximum index value attained during the life of the policy instead of the index value at maturity. It offers a minimum guaranteed return if the maximum index value is low. Even when the index has a high maximum value but drops substantially at maturity, the lookback EIA provides customers with higher interest than a point-to-point EIA. However, the option embedded in the lookback EIA is more expensive than the option in a point-to-point EIA. For this reason, the participation rate of the lookback EIA is lower than, the point-to-point EIA.

Heynen and Kat (1994a) suggest a way of reducing the price of a lookback option while preserving some of its good qualities. The solution, they say, lies in a partial lookback option with a floating strike that has a minimum and increases with the maximum value of an underlying asset.

The variable guaranteed return of the EIA will increase with the maximum index value for the monitoring period and will be bounded below by the minimum guaranteed return. If the index return multiplied by the participation rate is bigger than the variable guaranteed return, it will be the return credited to the policy. Otherwise, the EIAs will provide customers with the variable guaranteed return.

Let's take a closer look at the payoff of the partial lookback EIA as follows:

$$S(0)[\alpha(e^{X(T)} - 1) + 1], \text{ if } e^{X(T)} > \lambda e^{\max(M(t), L)}$$

$$S(0)[\alpha(\lambda e^{\max(M(t), L)} - 1) + 1], \text{ otherwise,} \quad (5.1)$$

this is,

$$S(0)[\alpha(\lambda e^{\max(M(t), L)} - e^{X(T)})_+ + \alpha(e^{X(T)} - 1) + 1], \quad (5.2)$$

where g denotes the minimum guaranteed return for the contract period, L denotes $\log[(g/\alpha+1)/\lambda]$, and λ lies between zero and one, and $t \leq T$. Note that the variable guaranteed return is $\alpha(\lambda e^{\max(M(t), L)} - 1)$.

By the fundamental theorem of asset pricing, it follows from the payoff (5.2) that the time-0 value of this partial lookback EIA is

$$S(0)\{\alpha\, e^{-rT}\mathrm{E}[(\lambda e^{\max(M(t), L)} - e^{X(T)})_+; h^*] + \alpha + e^{-rT}(1 - \alpha)\}. \quad (5.3)$$

Applying conditional expectation with respect to the random variables $X(t)$ and $M(t)$, we can obtain the time-0 value of the partial lookback option embedded in the EIA,

$$e^{-rT}\mathrm{E}[(\lambda e^{\max(L, M(t))} - e^{X(T)})_+; h^*]$$

$$= -\Phi(f_1)\,\Phi\left(-e_1 + \frac{\log\lambda}{\sigma\ T-t}\right)$$

$$+ \lambda^{2(\frac{\mu}{\sigma^2}+1)}\frac{\sigma^2}{2r}\Phi_2\left(d_1 + \frac{\log\lambda}{\sigma\ T}, -e_1 - \frac{\log\lambda}{\sigma\ T\text{-}t}; - \quad 1-\frac{t}{T}\right.$$

$$- \lambda\frac{\sigma^2}{2r}e^{-rT}\, e^{2(\frac{\mu}{\sigma^2}+1)L}\,\Phi_2\left(f_1 - \frac{2r\ \ t}{\sigma}, d_1 - \frac{2r\ \ T}{\sigma} + \frac{\log\lambda}{\sigma\ T}; \quad \frac{t.}{T}\right)$$

$$+ \lambda\left(1 + \frac{\sigma^2}{2r}\right) e^{-r(T\text{-}t)}\,\Phi(f_1)\,\Phi\left(-e_2 + \frac{\log\lambda}{\sigma\ T\text{-}t}\right)$$

$$+ \lambda e^{-rT}e^{L}\,\Phi_2\left(-d_2 + \frac{\log\lambda}{\sigma\ T}, -f_2; \quad \frac{t.}{T}\right)$$

$$-\Phi_2\left(-d_1 + \frac{\log\lambda}{\sigma\ T}, -f_1; \quad \frac{t.}{T}\right.$$

(5.4)

where for i = 1 and 2, d_i denotes $\dfrac{-L+(r+(-1)^{i-1}\,½\,\sigma^2)T,}{\sigma\ \ T}$